

2056

B.E. (Computer Science and Engineering)

Fourth Semester

CS-402: Linear Algebra and Probability Theory

Time allowed: 3 Hours

Max. Marks: 50

NOTE: Attempt five questions in all, including Question No. 1 which is compulsory and selecting two questions from each Part. Use of statistical Table is allowed. All questions carry 10 marks.

x-x-x

1. (a) Give any two practical situations where the problems are modeled by system of linear equations. Give one example of a non-homogeneous system whose solution set is empty.
- (b) Define diagonalizable matrices along with its applications. Give an example of a matrix that is not diagonalizable? $(5 \times 2 = 10)$
- (c) State the law of probability and derive it from the axioms of probability.
- (d) Explain how to find the marginal distribution of a continuous random variable from a joint probability density function $f(x, y)$. Also, state the condition that $f(x, y)$ must satisfy.
- (e) Define the covariance of two random variables X and Y . How does the sign of the covariance indicate the relationship between the variables?

PART-A

2. (a) Determine the values of k for which the system: $x - ky + z = 0$; $kx + 3y - kz = 0$; $3x + y - z = 0$ has (i) only trivial solution, (ii) non-trivial solution.
- (b) Prove that a spanning set can be reduced to a basis by removing vectors. Demonstrate with a set of four vectors in R^3 . $(4 + 3 + 3)$
- (c) Define vector space. Verify all vector space axioms for the set P_3 of polynomials of degree less than or equal to three. Will the polynomials of degree three only form a vector space? Justify.
3. (a) Let $W = \{(x, y, z, w) \in R^4 | x + y - z = 0 \text{ and } y + z + w = 0\}$. Prove that W is a subspace of R^4 . Find the basis and dimension of W . Extend the basis of W to a basis for the entire space R^4 .
- (b) Examine whether the matrix $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ is diagonalizable or not? Also, examine whether the geometric multiplicity of the eigenvalue matches or fails to match its algebraic multiplicity. Justify.

(2)

4.(a) Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$ be a linear transformation defined by

$T(x, y, z, w) = \{x+y+z, y+z+w, x-w\}$. Find the rank, nullity, and a basis for the kernel and image of T . Verify the Rank-Nullity Theorem.

(b) Let $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be a linear transformation defined by

$T(x, y, z) = \{3x, x-y, 2x+y+z\}$. Prove that T is an isomorphism. Also, find T^{-1} .

PART-B

5 (a) Define the variance and covariance of two random variables. Derive the formula for the variance of a linear combination of two random variables $Z = aX + bY$, where X and Y are random variables and a, b are constants. Let X and Y be two random variables with the properties: $E[X] = 2, E[Y] = 3, \text{Var}(X) = 4, \text{Var}(Y) = 9, \text{Cov}(X, Y) = 2$. Calculate the variance of the random variable $Z = 3X - 2Y$.

(b) State and prove Chebyshev's Inequality. A random variable has mean 50 and variance 25. Use Chebyshev's inequality to estimate: $P(40 < X < 60)$.

6. (a) Let X and Y have the following joint probability mass function:

$$P\{X = x, Y = y\} = \begin{cases} \frac{1}{6}, & \text{if } x = 1, 2, 3 \text{ and } y = 1, 2, 3 \\ 0, & \text{otherwise} \end{cases}$$

(i) Find the marginal distributions $P(X = x)$ and $P(Y = y)$.

(ii) Compute the conditional distribution $(P(Y = y) | X = 1)$.

(b) Define the binomial distribution and derive its mean and variance. Discuss the limiting case when n tends to infinity and p tends to zero, keeping $np = \lambda$ constant. Show how the binomial distribution approximates the Poisson distribution.

7. (a) Derive and explain the joint distribution of two independent random variables X and Y . Prove that their joint distribution can be written as the product of their marginal distributions.

(b) Let X and Y be two continuous random variables with the joint PDF:

$$f_{X,Y}(x, y) = \frac{1}{2}, 0 < x < 2, 0 < y < 2.$$

(i) Find the marginal distributions of X and Y , (ii) Compute the covariance between X and Y , (iii) Derive the conditional distribution of X given $Y = 1$.